

How much does it cost to teleport?

H. F. Chau and Hoi-Kwong Lo *

School of Natural Sciences, Institute for Advanced Study,
 Olden Lane, Princeton, NJ 08540

We show that the entropy of entanglement of a state characterizes its ability to teleport. In particular, in order to teleport faithfully an unknown quantum N -state, the two users must share an entangled state with at least $\log_2 N$ bits entropy of entanglement. We also note that the maximum capacity for a mixed state \mathcal{M} to teleport equals the maximum amount of entanglement entropy that can be distilled out from \mathcal{M} . Our result, therefore, provides an alternative interpretation for entanglement purification.

1 Introduction

In the process called quantum teleportation, an unknown quantum state is disassembled into, and later reconstructed from purely classical information and purely non-classical EPR correlations. Bennett *et al.* showed that two bits of classical information and one maximally entangled EPR pair are *sufficient* for the faithful teleportation of an unknown two-state quantum system [1]. They also showed how a simple modification of their method can be used to teleport an N -state object with the resources of $2\log_2 N$ bits of classical information and a pair of N -state particles in a completely entangled state shared by the two users. We shall briefly describe the teleportation scheme introduced by Bennett *et al.* in §2.

There are a number of conceptually important questions on quantum teleportation that we would like to answer. First, what are the minimal resources needed for a general teleportation scheme? A partial answer was given in Ref. [1] where Bennett *et al.* proved that the impossibility of superluminal communication implies that the reliable teleportation of an N -state would require a *classical* channel of $2\log_2 N$ bits. In spite of the recent advances in quantum information theory (see for example [4, 8, 12, 15]), the issue of the *minimal* amount of *quantum* sources required to teleport an N -state object has never been answered directly.¹ Using the idea of Hilbert space dimension counting, we give a simple proof that, for the faithful teleportation of an N -state object, the two users must share no less than $\log_2 N$ bits entropy of entanglement in §3. Consequently, the entropy of entanglement of a quantum state can also be interpreted as a measure of the usefulness of that state in teleportation.

The second question that we would like to address concerns the capability of faithful teleportation using mixed states. Owing to the interactions with the surroundings, the entangled quantum state shared by Alice and Bob before the teleportation should, in general, be a mixed state.

Faithful teleportation is nonetheless possible: Using the so-called entanglement purification scheme [2, 3, 4], one can distill out some maximally entangled states. These distilled states can then be used in faithful teleportation. In §4, we note that the maximum capability for a mixed state \mathcal{M} to teleport is equal to the maximum amount of entanglement entropy that can be distilled out from \mathcal{M} . Finally, a summary is given in §5.

2 Teleportation scheme of Bennett *et al.*

Teleportation is a method of indirectly sending a quantum state from one place to another. Conventionally, the sender is called Alice and the receiver is called Bob. As we shall discuss below, Alice sends Bob two messages: a quantum message at any time before the actual teleportation, and a classical message during the actual teleportation. Teleportation, as opposed to directly sending the quantum particle, is preferred when the quantum channel between Alice and Bob at the time of the quantum data transfer is jammed or noisy.

Let us first consider a simple example of teleporting a two-state particle. In order to teleport an unknown state in the form $\alpha|0\rangle + \beta|1\rangle$ from Alice to Bob, they perform the following operations [1]:

1. Alice prepares an EPR singlet. She sends one of the EPR particle to Bob through a quantum channel (and for the time being, we assume the channel is noiseless). She retains the second EPR particle for herself.
2. Alice makes sure that Bob has received the EPR particle.
3. Alice makes a joint measurement on the combined system of the EPR particle that she retains and the unknown quantum state that she wants to teleport along the four Bell basis, namely, $|\Psi^\pm\rangle = (|10\rangle \pm |01\rangle)/\sqrt{2}$ and $|\Phi^\pm\rangle = (|11\rangle \pm |00\rangle)/\sqrt{2}$.

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¹One may answer this question by using the theorem on p.14 of Ref [4], and we shall return to this point later in §3.

4. She tells Bob the result of her measurement via a classical communication channel (and once again, we assume the classical channel is noiseless).
5. Bob reconstructs the original unknown state by applying an unitary transformation U to his EPR particle according to the measurement result of Alice that he receives from the classical channel. In fact, $U = -I, -\sigma_3, \sigma_1$, and $i\sigma_2$ if the measurement result of Alice is $|\Psi^-\rangle, |\Psi^+\rangle, |\Phi^-\rangle$, and $|\Phi^+\rangle$ respectively, where σ_i 's are the Pauli spin matrices.

Because of the linearity of quantum mechanics, teleportation of n two-state particles can be achieved by teleporting the particles one by one. Besides, it is obvious that the above method can be used to teleport both pure and mixed states.

The above scheme can be generalized for the teleportation of an N -state particle using $2\log_2 N$ classical bits of communication plus $\log_2 N$ EPR pairs. But, can we use resources fewer than the above scheme? In Ref. [1], Bennett *et al.* argued that teleportation of an N state particle using less than $2\log_2 N$ classical bits of communication would violate causality. In the coming section, we show that at least $\log_2 N$ bits of entanglement entropy is also required. Thus, the Bennett *et al.* scheme is optimal.

3 Minimum quantum resources required in teleportation

We give the definition of entanglement entropy before discussing the amount of quantum resources needed for teleportation. Actually, there are a number of inequivalent definitions for the entanglement entropy of formation for a mixed quantum state. And here in definition 3, we use the one proposed by Bennett *et al.* in Ref. [4].

Definition 1: The entanglement entropy $E(|\Psi\rangle)$ of a pure state $|\Psi\rangle$ shared between two parties, Alice and Bob, is defined as the von Neumann entropy $S(\text{Tr}_{\text{Alice}}|\Psi\rangle\langle\Psi|)$ of the mixed state which it appears to Alice (or Bob) to be.

Definition 2: The entanglement entropy of an ensemble of pure states $\mathcal{E} = \{p_i, |\Psi_i\rangle\}$ shared between Alice and Bob is defined as the ensemble average $\sum_i p_i E(|\Psi_i\rangle)$ of the entanglement entropy of the pure states in the ensemble.

Definition 3: The entanglement entropy of formation of a mixed state \mathcal{M} is the minimum of $E(\mathcal{E})$ over ensembles \mathcal{E} realizing the mixed state \mathcal{M} .

The resources used by the two users, Alice and Bob, may be decomposed into two (namely classical and quantum) parts: In addition to some un-entangled states that

they may possess individually, they also share an entangled pair of M -state objects which may be completely or partially entangled. The idea of our proof on the minimum resources required in teleportation is very simple. First, we show that, for the faithful teleportation of an N -state object, M must be larger than or equal to N . Second, assuming that quantum teleportation of an N -state object can be achieved with less than $\log_2 N$ bits entropy of entanglement, we show that the condition $M \geq N$ will be violated. Therefore, to avoid contradiction, it must be the case that at least $\log_2 N$ bits entropy of entanglement are needed for teleporting an N -state object. An alternative proof based on the idea [2, 4] that local operations never increase the entropy of entanglement between Alice and Bob can be made. However, we stick to the following proof since it is conceptually simpler and interesting in its own right.

Lemma 1: In order to reliably teleport an N -state quantum object, Alice and Bob must share an entangled pair of M -state objects with $M \geq N$.

Proof: We prove by contradiction. Let us assume that Alice and Bob can succeed in reliably teleporting the state $|\phi\rangle$ of an unknown N -state object with an entangled pair of M -state objects where $M < N$. During the teleportation process, Bob must be able to reconstruct the state $|\phi\rangle$. Let $|\Psi_{\text{Bob}}\rangle$ denote the quantum state of the M -state particle in his share after Alice's measurement. Of course, the state $|\Psi_{\text{Bob}}\rangle$ depends on the outcome of Alice's measurement which is also communicated to Bob. In addition, Bob may process some auxiliary particles in a state denoted by $|\Psi_{\text{Aux}}\rangle$ which is independent of the result of Alice's measurement and the state $|\phi\rangle$ of the object to be teleported. Bob must then apply an unitary transformation U^{result} to $|\Psi_{\text{Bob}}\rangle \otimes |\Psi_{\text{Aux}}\rangle$ to reconstruct the state $|\phi\rangle$. The unitary operator U^{result} is a function of the result of Alice's measurement. But now for *any* Alice's measurement result, the support of Bob's constructed state, $U^{\text{result}}|\Psi_{\text{Bob}}\rangle \otimes |\Psi_{\text{Aux}}\rangle$, is at most M -dimensional. Since by assumption $M < N$, Bob will clearly fail in reconstructing the original state $|\phi\rangle$: If Bob were to succeed in such a reconstruction, transmission of any N -state quantum object could be decomposed into the transmission of a M -state quantum object ($M < N$) plus some classical bits! This is clearly impossible. \square

Theorem 1: In order to teleport an unknown N -state quantum object, Alice and Bob must share an entangled quantum state with entropy of entanglement $E \geq \log_2 N$ bits.

Proof: Let us assume that quantum teleportation of an unknown N -state object can be achieved with $E (< \log_2 N)$ bits entropy of entanglement shared between Alice and Bob. By separately applying quantum data compression to their respective subsystems, Alice and Bob could squeeze the original entanglement into a smaller number

of shared pairs of qubits [9, 14]. Using this two-sided compression for r shared pairs of entanglement E , Alice and Bob will each possess slightly more than rE qubits having slightly less than rE bits entropy of entanglement. The state of those slightly larger than rE qubits will be an excellent approximation of that of the original rM qubits and can be used for an almost faithful teleportation of an N^r -state object. But those slightly larger than rE qubits have a total Hilbert space dimension less than N^r and yet they are supposed to be sufficient for the reliable teleportation of an N^r -state object. This contradicts Lemma 1. \square

Consequently, if Alice uses n pairs of entangled qubits to teleport an unknown state of n qubits, the n pairs of qubits she has to prepare must be maximally entangled. In this respect, the teleportation scheme proposed by Bennett *et al.* through an ideal channel in Ref. [1] is optimal since it requires the least possible amount of entanglement entropy shared between Alice and Bob.

The following theorem tells us that the bound given in Theorem 1 is tight:

Theorem 2: Given a pure state $|\Psi\rangle$ shared between two parties. It can be used to teleport $E(|\Psi\rangle)$ qubits.

Proof: Given a sufficiently large number of copies of $|\Psi\rangle$, Alice and Bob can apply an entanglement purification scheme (details of the schemes can be found, for example, in Refs. [2, 3, 4]). Since $|\Psi\rangle$ is a pure state, the maximum number of EPR pairs that can be distilled out per copy of $|\Psi\rangle$ equals $E(|\Psi\rangle)$ [4, 15]. Then using the Bennett *et al.*'s teleportation scheme, we succeed in teleporting $E(|\Psi\rangle)$ qubits using $|\Psi\rangle$. \square

Thus, for a given pure state shared between two parties, the entanglement entropy of that state can be interpreted as a measure of the maximum capability of the state as an agent for teleportation. Nevertheless, the following example shows that knowing only the entanglement entropy of the quantum state share between Alice and Bob alone is not sufficient to carry out faithful teleportation.

Example 1: Consider the teleportation scheme proposed by Bennett *et al.* in Ref. [1]. But instead of using an EPR singlet, Alice and Bob share the state $|\Psi\rangle = (|11\rangle + |00\rangle)/\sqrt{2}$. Clearly the entropy of entanglement for $|\Psi\rangle$ equals one bit. Therefore, Theorem 1 tells us that it can be used to teleport one qubit from Alice to Bob. However, following the procedure of Bennett *et al.*, if Alice prepares a state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, then after the teleportation, Bob gets $|\psi'\rangle = -\beta|0\rangle + \alpha|1\rangle$, which is orthogonal to $|\psi\rangle$. The reason why this teleportation scheme fails completely is that Bob applies an incorrect unitary operation on his quantum particle after getting the classical message from Alice. In fact, the unitary operation that Bob needs to perform on his quantum particles depends on which entangled state Alice and

Bob share, not just the amount of entropy entanglement between them.

Further discussions on the conditions on the joint measurement needed for faithful teleportation can be found in Ref. [6].

4 Equivalence of the purification and teleportation capability

So far, our discussion is restricted to the case where the quantum state shared by Alice and Bob is pure. But in real life, decoherence occurs when the quantum particles are transmitted through a noisy channel. Thus, the quantum particles shared by Alice and Bob should be described by a mixed state. Given a mixed state \mathcal{M} , the two users are still able to perform faithful teleportation. Suppose Alice and Bob share a number of identical copies of the mixed states. By means of measurements on some of their quantum particles together with some classical communications between them, they can distill out a smaller set of quantum particles which are maximally entangled. Such schemes are called entangled purification protocols, and we refer to Refs. [2, 3, 4] for their detailed procedures. Now Alice and Bob carry out faithful teleportation using the purified maximally entangled states. In what follows, we prove that for a given mixed state \mathcal{M} , the above scheme for faithful teleportation is already the most efficient one. Now, we state some useful definition before giving our proof.

Definition 4: Let \mathcal{M} be a mixed state shared between Alice and Bob. $D_{A \rightarrow B}(\mathcal{M})$ denotes the maximum amount of entanglement entropy that can be distilled from \mathcal{M} by entanglement purification protocols which allow only one-way classical communications from Alice to Bob. $D_{B \rightarrow A}(\mathcal{M})$ is defined in a similar way. In addition, $D_{A \leftrightarrow B}(\mathcal{M})$ denotes the maximum amount of entanglement entropy that can be distilled from \mathcal{M} by entanglement purification protocols which allow two-way classical communications between both Alice and Bob.

Definition 5: Let \mathcal{M} be a mixed state shared between Alice and Bob. $T_{A \rightarrow B}(\mathcal{M})$ denotes the maximum amount of qubits that Alice can faithfully teleport, given that only one-way classical communications from Alice to Bob is allowed. Similarly, $T_{A \leftrightarrow B}(\mathcal{M})$ denotes the maximum amount of qubits that Alice can faithfully teleport when two-way classical communications between both Alice and Bob is allowed.

Remark 1: It is easy to see that $E(\mathcal{M}) \geq D_{A \leftrightarrow B}(\mathcal{M}) \geq D_{A \rightarrow B}(\mathcal{M})$ and $E(\mathcal{M}) \geq T_{A \leftrightarrow B}(\mathcal{M}) \geq T_{A \rightarrow B}(\mathcal{M})$ for any mixed state \mathcal{M} . And as shown in Ref. [4], there are situations where $D_{A \leftrightarrow B}$ is strictly greater than $D_{A \rightarrow B}$.

Theorem 3: $T_{A \rightarrow B} = D_{A \rightarrow B}$ and $T_{A \leftrightarrow B} = D_{A \leftrightarrow B}$.

Proof: We only prove the first equality. The proof of the second equality is similar to the first one. First, we show that $T_{A \rightarrow B} \geq D_{A \rightarrow B}$: For a given mixed state \mathcal{M} , we use the optimal purification scheme, which allows only classical communications from Alice to Bob, to give $D_{A \rightarrow B}(\mathcal{M})$ maximally entangled pairs per impure pair. Since the quantum state of these purified pairs are known, and they are maximally entangled, from Theorem 2 and Example 1, we can faithfully teleport $D_{A \rightarrow B}(\mathcal{M})$ qubits from Alice to Bob using only one way classical communication from Alice to Bob. Thus, $T_{A \rightarrow B} \geq D_{A \rightarrow B}$.

It remains to show that $D_{A \rightarrow B} \geq T_{A \rightarrow B}$. Again, we consider a given mixed state \mathcal{M} . By means of the optimal teleportation scheme involving only one way classical communication from Alice to Bob, we can faithfully teleport $T_{A \rightarrow B}(\mathcal{M})$ qubits per impure pair. Clearly, Alice can prepare some perfectly entangled EPR pairs, and then use the above teleportation scheme to faithfully “transport” half of her pairs to Bob. After this, Alice and Bob are able to share $T_{A \rightarrow B}(\mathcal{M})$ perfectly entangled EPR pairs per impure pair. Thus, $D_{A \rightarrow B} \geq T_{A \rightarrow B}$. \square

Theorem 3 provides an alternative interpretation for $T_{A \rightarrow B}$ and $T_{A \leftarrow B}$. They measure the capability of both faithful teleportation and entanglement purification using one-way and two-way classical communications, respectively.

Recently, using some ideas from teleportation, Bennett *et al.* argued that entanglement purification schemes are closely related to quantum error-correcting code (see for example Refs. [5, 7, 10, 11, 13, 16, 17] for the various quantum error-correcting codes proposed). And it is interesting to further investigate the relationship between teleportation and quantum error-correcting codes.

5 Summary

In summary, we study the cost of a general teleportation scheme. Using a simple idea of Hilbert space dimension counting, we prove in §3 that in order to teleport an unknown N -state quantum signal, a quantum state with entanglement entropy E of at least $\log_2 N$ is required to be shared between Alice and Bob. Consequently, we conclude that the Bennett *et al.*’s teleportation scheme via an ideal quantum channel is optimal because it uses the minimum possible amount of classical and quantum resources. We also argue that the entanglement entropy for a pure state can be interpreted as the usefulness of a state in teleportation.

We go on to consider the case of mixed state. we find that the maximum capability of a mixed state \mathcal{M} to perform faithful teleportation is equal to the maximum amount of entanglement entropy that can be distilled out

from \mathcal{M} . This provides, once again, an alternative interpretation of $D_{A \rightarrow B}$ and $D_{A \leftarrow B}$.

A number of open questions remain. First, what is the relation between entanglement entropy of formation and the maximum amount of qubits that a mixed state can faithfully teleport? It is quite conceivable that $E(\mathcal{M}) > T_{A \leftarrow B}(\mathcal{M})$ for some mixed state \mathcal{M} , although a rigorous proof is lacking (compare with Remark 1). Second, can we characterize the fidelity of teleportation when the entangled state shared between the two parties actually differs slightly from the one they have in mind?

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